The homology of L-fuzzy simplicial complexes

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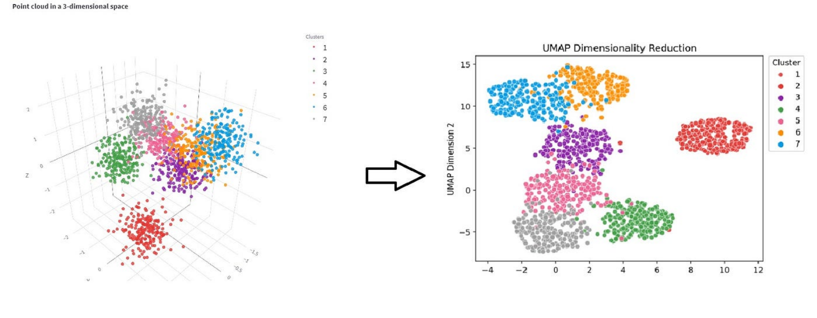
The inspiration: UMAP

UMAP: **Uniform Manifold Approximation and Projection**

[McInnes et al., 2018]



The inspiration: UMAP

- The paper models the high-dimensional dataset as a

  fuzzy simplicial complex

  before the projection.
- However, in practice, the **UMAP** algorithm does not really use the available *fuzzy* information.
- This leaves us with two open questions:

The inspiration: UMAP

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- However, in practice, the **UMAP** algorithm does not really use the available *fuzzy* information.
- This leaves us with two open questions:

What information can be extracted from a fuzzy simplicial complex?

What would its homology be like?

Background

The closest work we have found is this article:

Singular homology groups of fuzzy topological spaces
[Wang-jin and Chong-you, 1985]

Disadvantages:

- It does not exactly match our question.
- Impossible to translate into a computational framework.

Background

We saw that no one had defined the homology of a fuzzy simplicial complex. So, what do we do?

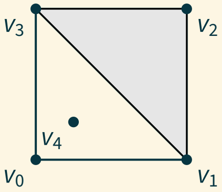
We make it up ourselves!

But first, we need to define:

- What simplicial homology is and how it is computed.
- What fuzzy sets and L-fuzzy sets are.

Simplicial Homology

We start with a simplicial complex Δ:



Let R be a principal ideal domain, and define the chain complex:

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to$$

where the boundary operator ∂_d maps each d-simplex to the $(d-1)$-simplices on its boundary.

Simplicial Homology

The boundary operator ∂_d can be encoded as a matrix M_d.

We define the following submodules of C_d:

- **Cycles submodule:** $Z_d = \ker(\partial_d) \subseteq C_d$
- **Boundaries submodule:** $B_d = \mathrm{Im}(\partial_{d+1}) \subseteq C_d$

Since $\partial^2 = 0$, we have $B_d \subseteq Z_d$ and can define:

- **Homology of Δ:** $H_d = \frac{Z_d}{B_d}$

Simplicial Homology: Summary

To compute homology we need:

- A simplicial complex
- R-modules
- Kernel and image of a homomorphism between R-modules
- Quotient of R-modules

Simplicial Homology: Summary

To compute ***L*-fuzzy** homology we need:

- An ***L*-fuzzy** simplicial complex
- ***L*-fuzzy** R-modules
- Kernel and image of a homomorphism between ***L*-fuzzy** R-modules
- Quotient of ***L*-fuzzy** R-modules

So, we are going to define the L-fuzzy versions of all these concepts.

Lattices

A ***completely distributive lattice (CDL)*** is a poset $(L, \leqslant)$ where:

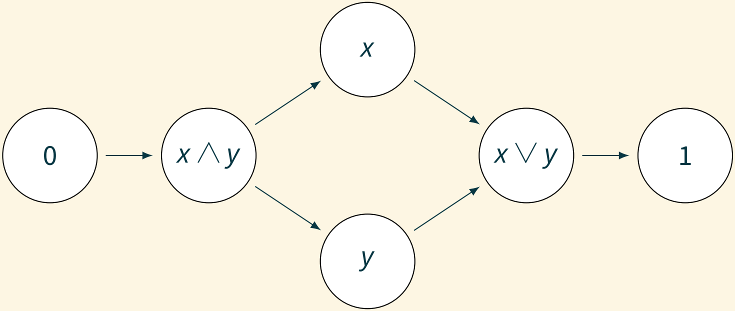
- Every $S \subseteq L$ has a ***meet*** $\bigwedge S \in L$, which is the greatest lower bound of S.
- Every $S \subseteq L$ has a ***join*** $\bigvee S \in L$, which is the least upper bound of S.
- There is a global minimum and maximum: $0, 1 \in L$.
- The meet and join are ***completely distributive***:

$$\bigwedge_i \bigvee_j l_{ij} = \bigvee_{f \in J^I} \bigwedge_i l_{if(i)} \quad \text{and} \quad \bigvee_i \bigwedge_j l_{ij} = \bigwedge_{f \in J^I} \bigvee_i l_{if(i)}.$$

Lattices

Examples of CDLs

- Totally ordered sets such as $\{0, 1\}$ and $[0, 1]$ are CDLs.
- The free distributive lattice $\boldsymbol{FDL}(x, y)$:



L-fuzzy sets

Definition

Given a set X and a CDL L, an ***L-fuzzy subset*** is a map $\mu : X \to L$.

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- The value $\mu(x) \in L$ represents the degree of membership of x respect to the L-fuzzy subset:

$\mu(x) = 0$	$0 < \mu(x) < 1$	$\mu(x) = 1$
x is not a member	x is partially a member	x is a member

- We say that $\mu \subseteq \nu$ if $\mu(x) \leqslant \nu(x)\ \forall x \in X$.
- When $L = \{0, 1\}$, we are just describing the classical (crisp) subsets of X.
- The first definition from [Zadeh, 1965] only considers $L = [0, 1]$.

L-fuzzy simplicial complexes

Definition

Given a simplicial complex Δ and a CDL L, an ***L-fuzzy subcomplex*** is a map $\mu : \Delta \to L$ such that:

$$\text{if } \sigma \subseteq \tau, \text{ then } \mu(\sigma) \geqslant \mu(\tau).$$

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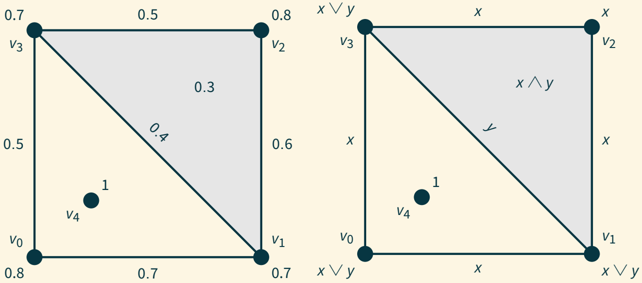
Two L-fuzzy simplicial complexes with $L = [0, 1]$ and $L = \mathrm{FDL}(x, y)$:



L-fuzzy modules

Definition

Given an R-module M and a CDL L, an ***L-fuzzy submodule*** is a map $\mu : M \to L$ such that:

$$\mu(0) = 1, \quad \mu(x + y) \geqslant \mu(x) \wedge \mu(y) \quad \text{and} \quad \mu(r \cdot x) \geqslant \mu(x).$$

Extension principle

Definition (Extension Principle)

Let $f : X \to Y$ be a map, and let $\mu : X \to L$ and $\nu : Y \to L$ be L-fuzzy subsets.

- The ***image*** of μ under f is the L-fuzzy subset $f(\mu) : Y \to L$ defined by

$$f(\mu)(y) \;=\; \bigvee\{\, \mu(x) \mid f(x) = y \,\}.$$

- The ***preimage*** of ν under f is the L-fuzzy subset $f^{-1}(\nu) : X \to L$ defined by

$$f^{-1}(\nu)(x) \;=\; \nu\big(f(x)\big).$$

Quotient of L-fuzzy submodules

Definition

Let $\mu, \nu : M \to L$ be two L-fuzzy submodules such that $\mu \subseteq \nu$.
Then the ***quotient of ν with respect to μ*** is an L-fuzzy submodule $(\nu/\mu) : M/\langle \mu^* \rangle \to L$ defined by:

$$\left(\frac{\nu}{\mu}\right)([x]) = \bigvee \{\nu(z) \mid z \in [x]\},$$

where $x \in \langle \nu^* \rangle$ and $[x]$ denotes the coset $x + \langle \mu^* \rangle$.

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Remark

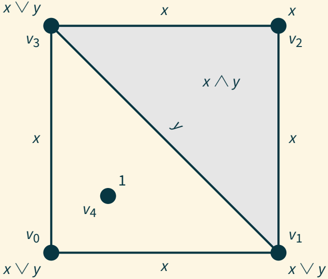
We already have all the ingredients to define the L-fuzzy version of simplicial homology. ***Let's do it!***

L-fuzzy simplicial homology

Definition

Let Δ be a simplicial complex, $\mu : \Delta \to L$ an L-fuzzy subcomplex and C_d the R-module of d-chains generated by $\Delta_d = \{\sigma_{d,1}, \ldots, \sigma_{d,N_d}\}$.

- We define the ***L-fuzzy subset of d-simplices*** $\delta_d : C_d \to L$ as: $\delta_d = \bigcup_{i=1}^{N_d} (\sigma_{d,i})_{\mu(\sigma_{d,i})}$



$$\delta_1(c) = \begin{cases} x & \text{if } c = [v_0, v_1], \\ x & \text{if } c = [v_0, v_3], \\ x & \text{if } c = [v_1, v_2], \\ y & \text{if } c = [v_1, v_3], \\ x & \text{if } c = [v_2, v_3], \\ 0 & \text{for any other 1-chain} \end{cases}$$

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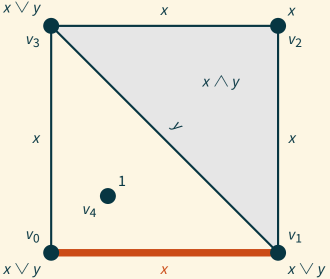
- We define the ***L-fuzzy submodule of d-chains*** $\kappa_d : C_d \to L$ as: $\kappa_d = \langle \delta_d \rangle$.



$c_1 = [v_0, v_1]$

$$\begin{cases} \kappa_1(0) = & 1 \\ \kappa_1(c_1) = & x \\ & \end{cases}$$

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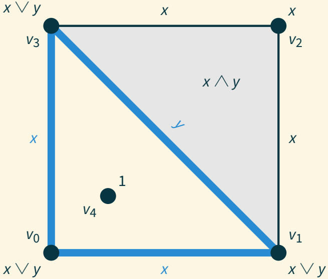

$$c_2 = [v_0, v_1] + [v_1, v_3] - [v_0, v_3]$$

$$\begin{cases} \kappa_1(0) = & 1 \\ \kappa_1(c_1) = & x \\ \kappa_1(c_2) = & x \wedge y \end{cases}$$

L-fuzzy simplicial homology

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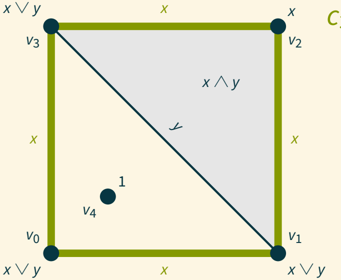


$$c_3 = [v_0, v_1] + [v_1, v_2] + [v_2, v_3] - [v_0, v_3]$$

$$\begin{cases} \kappa_1(0) = & 1 \\ \kappa_1(c_1) = & x \\ \kappa_1(c_2) = & x \wedge y \\ \kappa_1(c_3) = & x \\ \vdots & \vdots \end{cases}$$

L-fuzzy simplicial homology

Definition

Let Δ be a simplicial complex, $\mu : \Delta \to L$ an L-fuzzy subcomplex and C_d the R-module of d-chains generated by $\Delta_d = \{\sigma_{d,1}, \ldots, \sigma_{d,N_d}\}$.

- We define the ***L-fuzzy submodule of d-cycles*** $\zeta_d : C_d \to L$ as: $\zeta_d = \kappa_d \cap \partial_d^{-1}(0_1) = \kappa_d \cap (\ker \partial_d)_1 \in \mathfrak{FM}(C_d, L)$.
- We define the ***L-fuzzy submodule of d-boundaries*** $\beta_d : C_d \to L$ as: $\beta_d = \kappa_d \cap \partial_{d+1}(\kappa_{d+1}) \in \mathfrak{FM}(C_d, L)$.

Proposition

$$\beta_d \subseteq \zeta_d \subseteq \kappa_d$$

L-fuzzy simplicial homology

Definition

Let $\mu : \Delta \to L$ be an L-fuzzy subcomplex. We define the ***L-fuzzy d-th homology*** submodule of μ as:

$$\eta_d = \frac{\zeta_d\big|_{\langle \zeta_d^* \rangle}}{\beta_d\big|_{\langle \zeta_d^* \rangle}} : \frac{\langle \zeta_d^* \rangle}{\langle \beta_d^* \rangle} \to L.$$

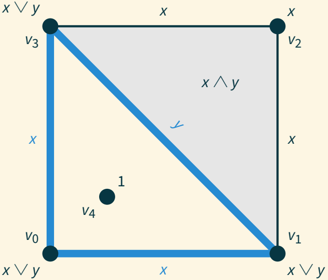
Remark

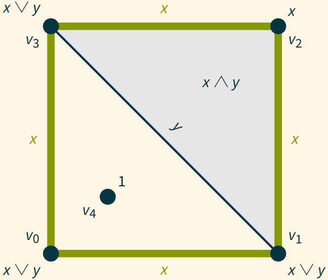
If $\mu^* = \Delta$ and 0 is meet-prime in L, then $\eta_d : H_d \to L$ and

$$\eta_d([h]) = \bigvee \{ \kappa_d(c) \mid c \in [h] \}.$$

L-fuzzy simplicial homology

Consider the homology class $[c_2]$.





$c_2 \in [c_2]$ and $\kappa_1(c_2) = x \wedge y$.

$c_3 \in [c_2]$ and $\kappa_1(c_3) = x$.

Any other cycle $c \in [c_2]$ satisfies $\kappa_1(c) = x \wedge y$. Then,

$$\eta_1([c_2]) = (x \wedge y) \vee x = x.$$

L-fuzzy simplicial homology

For each lattice value $l \in L$ there is a level submodule:

$$\eta_d^{\geqslant l} = \{[h] \in H_d \mid \eta_d([h]) \geqslant l\} \subseteq H_d$$

The submodules $\eta_d^{\geqslant l}$ form a lattice which fully describes η_d.

How can we compute the level submodules $\eta_d^{\geqslant l}$***?***

L-fuzzy simplicial homology

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The submodules $\eta_d^{\geqslant l}$ form a lattice which fully describes η_d.

How can we compute the level submodules $\eta_d^{\geqslant l}$?

The general idea is:

- While computing H_d, we reduce M_d to its SNF M'_d.
- We get the change-of-basis matrix $M_d^{H,\sigma}$.
- We use $M_d^{H,\sigma}$ to state some equations systems which help us to compute $\eta_d([h])$ and $\eta_d^{\geqslant l}$.

This is ***always valid***, for any coefficient ring R and any lattice L.

The case with field coefficients

Consider now coefficients in a field F.
We want to describe the L-fuzzy homology with a simple invariant.

Definition

Let V be an F-vector space, let $\mu : V \to L$ be an L-fuzzy subspace and let $B = \{v_1, \ldots, v_n\}$ be a basis of V. We say that B ***generates*** μ if:

$$\mu(r_1 v_1 + \cdots + r_n v_n) = \bigwedge_{\substack{i=1,\ldots,n \\ r_i \neq 0}} \mu(v_i).$$

In that case, $\mu(B) = \{\mu(v_i) \mid i = 1, \ldots, n\}$ is the ***fuzzy multiset of*** B.

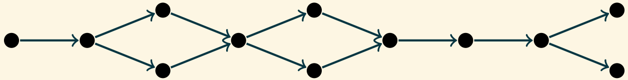
Does such a basis always exist?

The case with field coefficients

Definition

Let L be a CDL. We say that $S \subseteq L$ is a ***semi-chain*** if every value in S is incomparable to at most another value in S.

In general, a semi-chain looks like this:



Example

Any totally ordered set is a semi-chain.

Example

$\mathrm{FDL}(x, y)$ is a semi-chain.

The case with field coefficients

Proposition

Let V be an F-vector space and let $\mu : V \to L$ be an L-fuzzy subspace. Suppose that $\{\mu(u) \mid u \in V\} \subseteq L$ is a semi-chain.
Then, there exists a basis B of V that generates μ. Moreover, if another basis B' generates μ too, then $\mu(B) = \mu(B')$.

In this case, the multiset $\mu(B)$ is independent of the generating basis B, and we define the ***L-fuzzy rank*** of μ as $\mathrm{Rank}(\mu) = \mu(B)$.

The case with field coefficients

Let $\mu : \Delta \to L$ be an L-fuzzy simplicial subcomplex such that $\{\mu(\sigma) \mid \sigma \in \Delta\}$ is a semi-chain. Then the sets

$$\{\kappa_d(c) \mid c \in C_d\} \qquad \text{and} \qquad \{\eta_d(e) \mid e \in H_d\}$$

are semi-chains as well, and $\mathrm{Rank}(\eta_d)$ is well defined.

How can we compute $\mathrm{Rank}(\eta_d)$ ***?***

The case with field coefficients

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are semi-chains as well, and $\mathrm{Rank}(\eta_d)$ is well defined.

How can we compute $\mathrm{Rank}(\eta_d)$ ***?***

By applying zigzag homology.

The case with field coefficients

We will explain how to compute $\mathrm{Rank}(\eta_d)$ when $L = \mathrm{FDL}(x, y)$, and the same idea will apply to any semi-chain.

We start with the semi-chain $\mathrm{FDL}(x, y)$ ($a \to b$ means $a \leqslant b$):

$$0 \to x \wedge y \begin{smallmatrix} \nearrow & x & \searrow \\ \searrow & y & \nearrow \end{smallmatrix} x \vee y \to 1$$

We “split it in half”, getting the sequence:

$$1 \leftarrow x \vee y \leftarrow x \leftarrow x \wedge y \leftarrow 0 \to x \wedge y \to y \to x \vee y \to 1$$

The case with field coefficients

Denoting $\Delta^l = \mu^{\geqslant l}$, we get the zigzag diagram:

$$\Delta^1 \subseteq \Delta^{x \vee y} \subseteq \Delta^x \subseteq \Delta^{x \wedge y} \subseteq \Delta^0 \supseteq \Delta^{x \wedge y} \supseteq \Delta^y \supseteq \Delta^{x \vee y} \supseteq \Delta^1.$$

This induces the zigzag homology module:

$$\mathbb{H}_d : H_d^1 \to H_d^{x \vee y} \to H_d^x \to H_d^{x \wedge y} \to H_d^0 \leftarrow H_d^{x \wedge y} \leftarrow H_d^y \leftarrow H_d^{x \vee y} \leftarrow H_d^1,$$

whose barcode is

$$\mathsf{B}(\mathbb{H}_d) = \{[a_j, b_j] \mid j = 1, \ldots, N\}.$$

We must interpret the interval $[x, y]$ as $\{x, x \wedge y, 0, x \wedge y, y\}$.

The case with field coefficients

Proposition

Let $\mu : \Delta \to \mathrm{FDL}(x, y)$ an L-fuzzy subcomplex, and let $\eta_d : H_d \to \mathrm{FDL}(x, y)$ be the associated L-fuzzy homology submodule. Then,

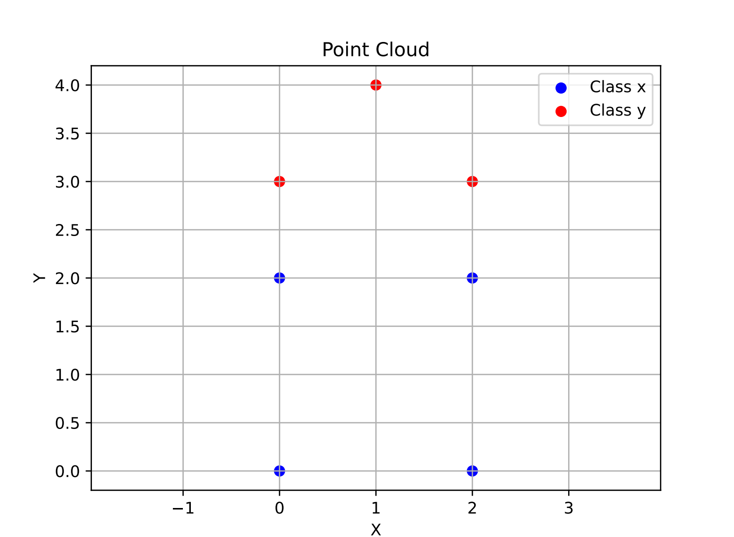
$$\mathrm{Rank}(\eta_d) = \{\, a_j \vee b_j \mid [a_j, b_j] \in \mathrm{B}(\mathbb{H}_d) \text{ and } 0 \in [a_j, b_j] \,\}.$$

The same statement holds for any semi-chain.

Let's see how to apply this proposition.

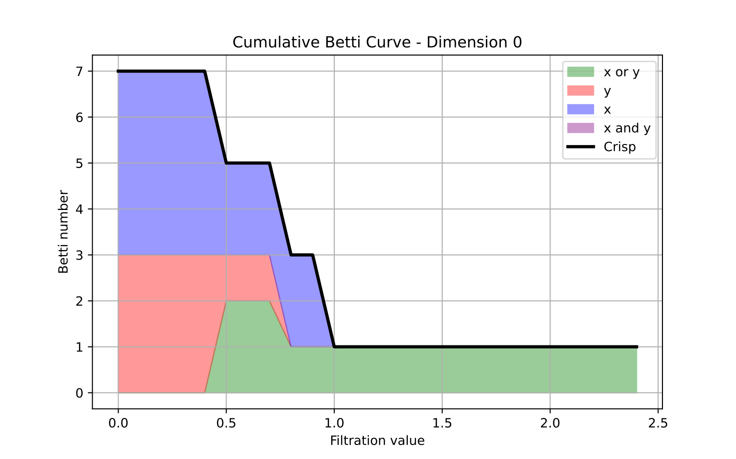
L-fuzzy Betti curve

Consider this bichromatic dataset and a filtration over it.
We can model it as an L-fuzzy dataset with $L = \mathrm{FDL}(x, y)$.



L-fuzzy Betti curve

The last Proposition gives us a sequence of L-fuzzy ranks, which results in the ***L-fuzzy Betti curve of η_d***.



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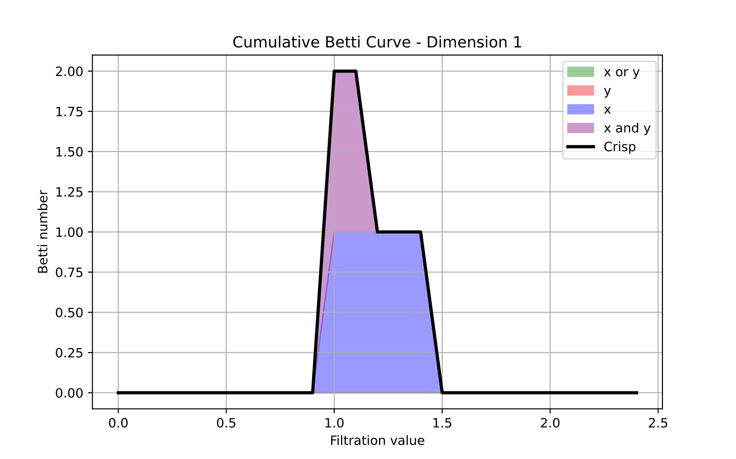
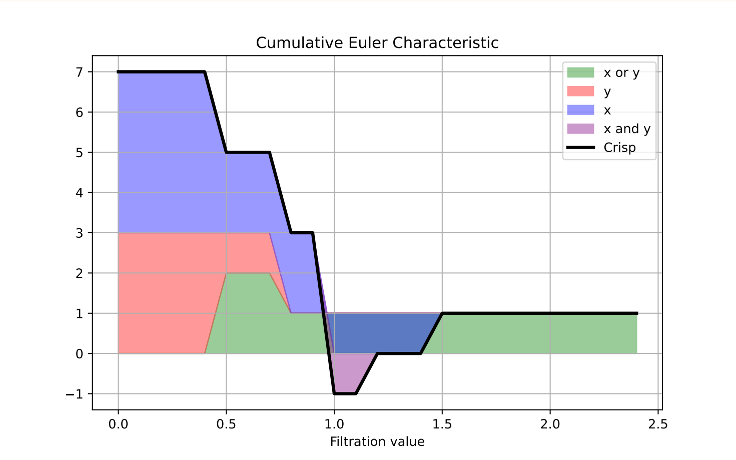


L-fuzzy Betti curve

We can also compute the ***L*-fuzzy Euler characteristic of** μ.



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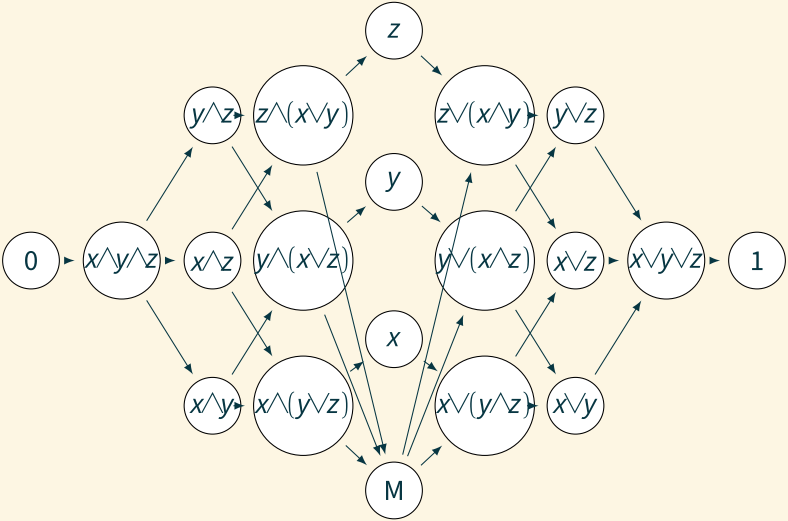
  Information and control, 8(3):338–353.

Appendix: Lattices

- The free distributive lattice ***FDL***(x, y, z):



Appendix: The case with field coefficients

Example

If $L = \{0, 1\}$, any $\mu \in \mathcal{FM}(V, L)$ is simply a crisp subspace $E \subseteq V$.
If $\dim(E) = m \leqslant n$, then

$$\mathrm{Rank}(\mu) = \left\{ \overbrace{1, \ldots, 1}^{m}, \overbrace{0, \ldots, 0}^{n-m} \right\}.$$

Appendix: The case with field coefficients

Example

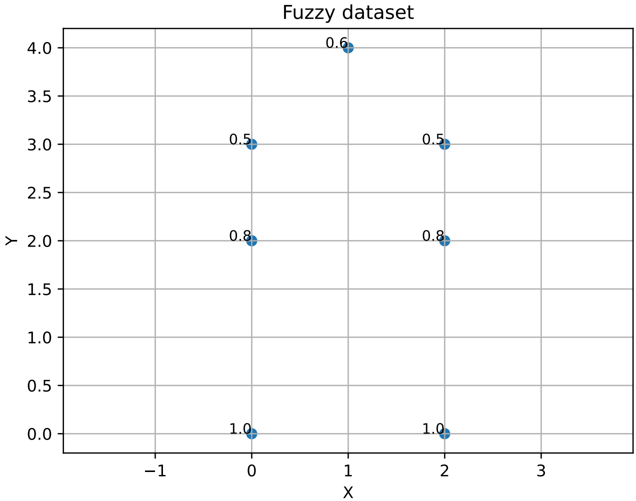
Let $V = \mathbb{R}^2$ and consider $\mu \in \mathcal{FM}(V, \mathrm{FDL}(x, y, z))$ defined by:

$$\mu((r_1, r_2)) = \begin{cases} 1 & \text{if } r_1 = r_2 = 0, \\ x \wedge y & \text{if } r_1 = r_2 \neq 0, \\ x \wedge z & \text{if } r_1 = 0 \text{ and } r_2 \neq 0, \\ y \wedge z & \text{if } r_2 = 0 \text{ and } r_1 \neq 0, \\ x \wedge y \wedge z & \text{otherwise.} \end{cases}$$

$\{\mu(u) \mid u \in V\}$ is not a semi-chain because $x \wedge y$, $x \wedge z$ and $y \wedge z$ are mutually incomparable, and there is no basis B that generates μ.

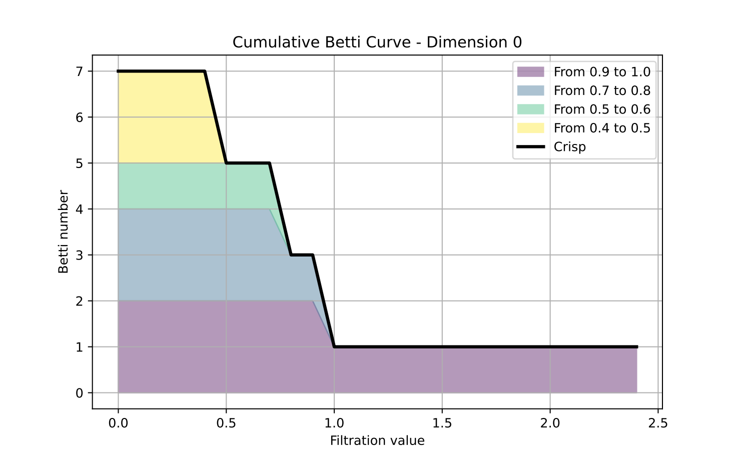
Appendix: *L*-fuzzy Betti curve

Consider a fuzzy dataset like this and build a filtration over it. We can apply this theorem on each complex in the filtration.



Appendix: L-fuzzy Betti curve

This gives us a sequence of L-fuzzy ranks, which results in the ***L-fuzzy Betti curve of η_d***.



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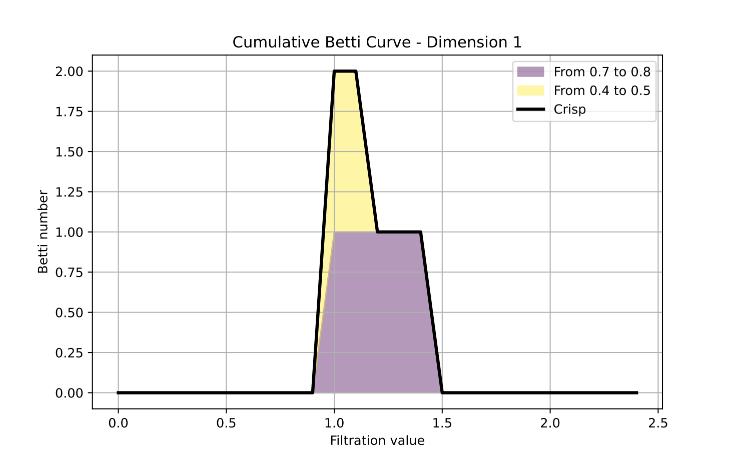

Appendix: L-fuzzy Betti curve

We can also compute the ***L-fuzzy Euler characteristic of*** μ.