The Induced Matching Distance: A Novel Topological Metric with Applications in Robotics

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Vietoris-Rips filtrations

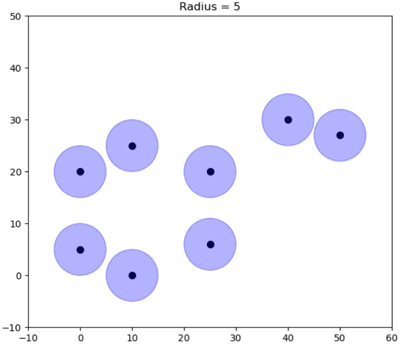
Consider a set of points $Z = \{z_1, z_2, \ldots, z_n\}$ with a symmetric non-negative function $d_Z \colon Z \times Z \to \mathbb{R}^+$.

Given $r \geq 0$, the 1-skeleton of the Vietoris-Rips complex is a graph $\mathrm{VR}_r(Z)$ such that:

- Z is the vertex set
- $[z, z']$ is an edge if and only if $d_Z(z, z') \leq r$

The Vietoris-Rips filtration of Z is $\mathrm{VR}(Z) = \{\mathrm{VR}_r(Z)\}_{r \in \mathbb{R}^+}$

Vietoris-Rips filtrations



Note that:

$$r \leq s \Rightarrow \mathrm{VR}_r(Z) \subseteq \mathrm{VR}_s(Z)$$

0-persistent homology

Given $r \geq 0$, the 0-homology group $\mathrm{H}_0(\mathrm{VR}_r(Z))$ is the free $\mathbb{Z}_2$-vector space generated by the connected components of $\mathrm{VR}_r(Z)$.

The 0-persistent homology of $\mathrm{VR}(Z)$, denoted $\mathrm{PH}_0(Z)$, is composed by:

- The set of 0-homology groups $\{\mathrm{H}_0(\mathrm{VR}_r(Z))\}_{r\in\mathbb{R}^+}$
- Linear maps $\left\{\rho^Z_{rs} \colon \mathrm{H}_0(\mathrm{VR}_r(Z)) \to \mathrm{H}_0(\mathrm{VR}_s(Z))\right\}_{r\leq s}$ induced by the inclusions $\mathrm{VR}_r(Z) \subseteq \mathrm{VR}_s(Z)$.

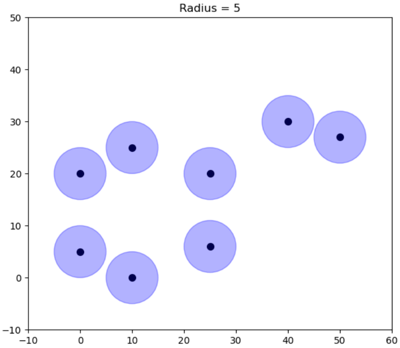
0-persistent homology

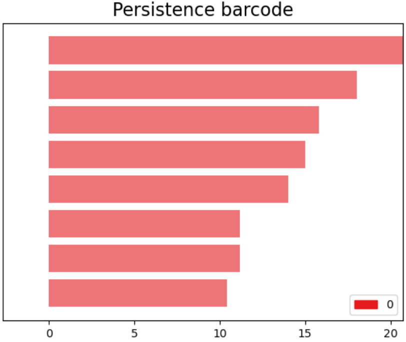
$\mathrm{PH}_0(Z)$ tracks the life of connected components in $\mathrm{VR}(Z)$.

- The items in $\mathrm{PH}_0(Z)$, called *classes*, are born at 0 since $\mathrm{VR}_0(Z)$ is the set of vertices with no edges.
- As r increases, some classes merge with others and die. A class $[z_j] \in \pi_0(\mathrm{VR}_0(Z))$ *dies at* $b > 0$ if:
  1) $\rho^Z_{0\ell}([z_j]) = [z_j]$ for all $\ell \in \mathbb{R}^+$ with $\ell < b$.
  2) $\rho^Z_{0b}([z_j]) = [z_i]$, for some $i < j$. That is, $[z_j] + [z_i] \in \ker \rho^Z_{0b}$.

Barcodes

Birth and deaths in $\mathrm{PH}_0(Z)$ are stored in a barcode $\mathrm{B}(Z) = (S^Z, m^Z)$ with representation $\mathrm{Rep}\ \mathrm{B}(Z) = \{\ (b, \ell) \mid b \in S^Z$ and $\ell \in \{1, 2, \ldots, m^Z(b)\}\ \}$.





Comparing barcodes

Now consider another set of points $X = \{x_1, x_2, \ldots, x_n\}$ with a symmetric non-negative function $d_X : X \times X \to \mathbb{R}^+$ and a bijection:

$$\begin{array}{rccc} f_\bullet : & X & \to & Z \\ & x_i & \mapsto & z_i \end{array}$$

We want to find the difference between $\mathrm{B}(X)$ and $\mathrm{B}(Z)$.

Comparing barcodes

A classical method is the q-Wasserstein distance:

$$W_q(\mathrm{B}(X), \mathrm{B}(Z)) = \inf_{\mu \in M} \left(\sum_{\substack{(a,\ell) \in \mathrm{Rep}\, B(X) \\ \mu((a,\ell)) = (b,\ell')}} |a - b|^q \right)^{1/q},$$

M is the set of all partial matchings $\mu \colon \mathrm{Rep}\,\mathrm{B}(X) \nrightarrow \mathrm{Rep}\,\mathrm{B}(Z)$.

Comparing barcodes

W_q compares all the possible partial matchings in M and uses the optimal one.

However, the bijection $f_\bullet \colon X \to Z$ induces an isomorphism:

$$f_0 \colon \mathrm{H}_0(\mathrm{VR}_0(X)) \to \mathrm{H}_0(\mathrm{VR}_0(Z))$$

and therefore a specific partial matching $\sigma_f^0 \in M$.

Induced matching distance

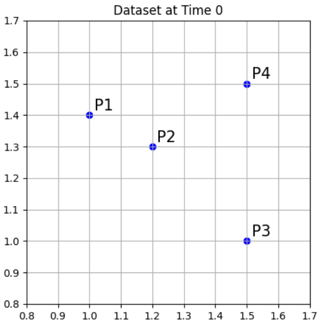
Then, we propose the q-induced matching distance:

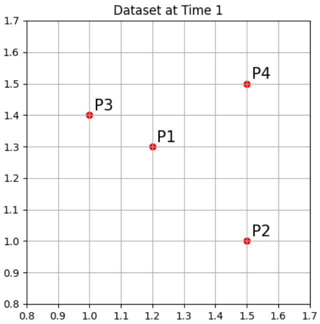
$$d_{f_0}^q(\mathrm{B}(X), \mathrm{B}(Z)) = \left(\sum_{\substack{(a,\ell)\in \mathrm{Rep}\, B(X) \\ \sigma_f^0((a,\ell))=(b,\ell')}} |a-b|^q \right)^{1/q}$$

Clearly, $W_q(\mathrm{B}(X), \mathrm{B}(Z)) \leq d_{f_0}^q(\mathrm{B}(X), \mathrm{B}(Z))$

Induced matching distance

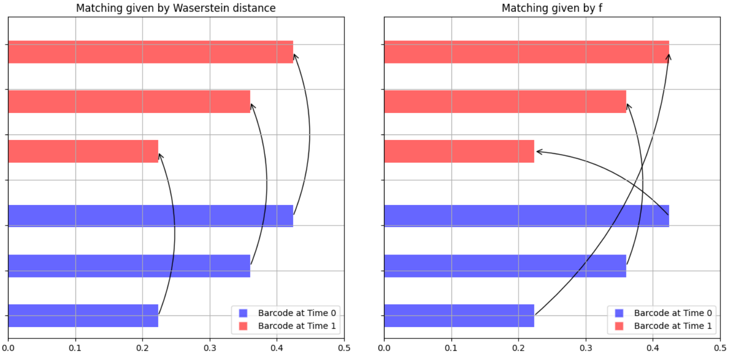
Let X_0 be the set of points $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3, \mathsf{P}_4$ at time 0.
Let X_1 be the set of points $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3, \mathsf{P}_4$ at time 1.
The bijection $f_\bullet \colon X_0 \to X_1$ is the trivial one, $f_\bullet(\mathsf{P}_i) = \mathsf{P}_i$.





Induced matching distance

These are the partial matchings that define the distances $W_q(\mathrm{B}(X_0), \mathrm{B}(X_1))$ and $d^q_{f_0}(\mathrm{B}(X_0), \mathrm{B}(X_1))$



Application: Robot fleet navigation analysis

As an application, we aim to use $d^q_{f_0}$ to compare three local navigation algorithms or *behaviors* for robots:

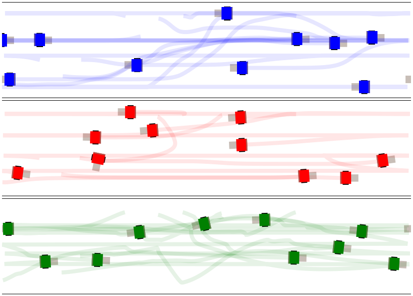
- Human-Like (HL)
- Optimal Reciprocal Collision Avoidance (ORCA)
- Social Force (SF)

Navground

We use Navground, a Python simulator for robots navigation.

Corridor scenario:

- 15m long, 3.5m wide, both ends connected.
- 10 agents with 0.8m of diameter and 1.2m/s of optimal speed.
- 5 agents driving left, 5 agents driving right.



We run 200 simulations with 900 steps for each behavior type.

Induced matching signal

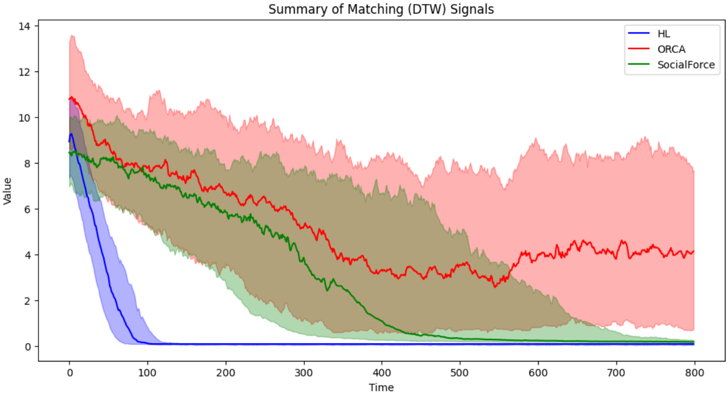
Given a simulation, we apply the following steps:

1. For $i = 1, \ldots, 10$, Agent $i \longrightarrow a^i = \left\{a_t^i = (x_t^i, y_t^i, \alpha_t^i)\right\}_{t=1}^{900}$
2. For $t = 1, 2, \ldots, 850$, $Z_t = \left\{z_t^i = \{a_t^i, a_{t+10}^i, \ldots, a_{t+50}^i\}\right\}_{i=1}^{10}$
3. DTW as distance $\longrightarrow \left\{\mathrm{VR}_0(Z_t)\right\}_{t=1}^{850} \longrightarrow \left\{\mathrm{B}(Z_t)\right\}_{t=1}^{850}$
4. For $t = 1, \ldots, 800$,
   $f_\bullet^t \colon Z_t \to Z_{t+50} \longrightarrow m = \left\{d^1_{f_0^t}(B(Z_t), B(Z_{t+50}))\right\}_{t=1}^{800}$

m is called the induced matching signal of the simulation

Induced matching signal

We get 600 induced matching signals, 200 for each behavior.



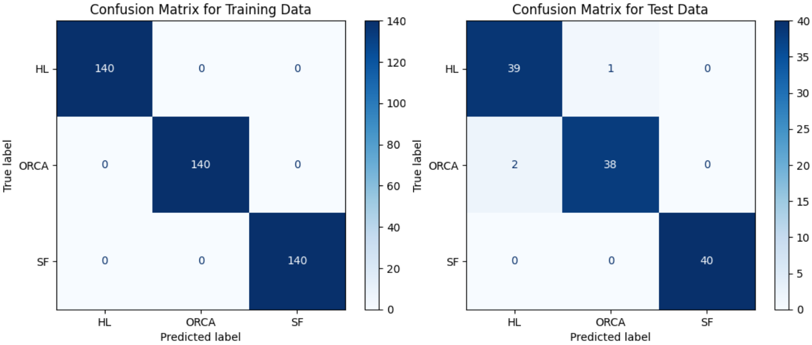
ResNet time series classifier

Induced matching signals can be used to predict the behavior.

- ResNet classifier for time series
- 420 signals for training
- 60 signals for validation
- 120 signals for testing

ResNet time series classifier

We got almost perfect results:



Code of the experiments:

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