A study on persistent homology with integer coefficients

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Abstract.

Homology is a tool that assigns to a topological space an abelian group for each dimension. Persistent homology is a more modern technique used to analyze the evolution of homology in a topological space that is built step by step. Both homology groups and persistent homology may be different according to the set of coefficients used in the calculations. Persistent homology with coefficients over a field such as $\mathbb{R}$ or $\mathbb{Z}_2$ has been widely studied and applied thanks to it easy computation, its stability and the existence of easy and complete invariants for its classification. Unfortunately, persistent homology over coefficients in $\mathbb{Z}$ is harder to calculate and classify, and because of that it has been barely studied. In [1], we start from the approach and concepts defined in [2] and we give some new results, we generalize the definitions and we give a partial stability result.

Let us introduce the details of [1]. The first step is to consider a topological space built on m steps, that is, an increasing sequence of m spaces. For a given dimension n, each space has its own homology groups, so we can see persistent homology as a diagram

$$0 = H_0 \xrightarrow{\rho_{0,1}} H_1 \xrightarrow{\rho_{1,2}} H_2 \xrightarrow{\rho_{2,3}} \cdots \xrightarrow{\rho_{m-1,m}} H_m \tag{1}$$

where H_i is the homology group in step i and $\rho_{i,j} = \rho_{j-1,j} \circ \cdots \circ \rho_{i,i+1}$ are group homomorphisms. From this diagram, the authors of [2] define the groups $H_{i,j} = \operatorname{Im} \rho_{i,j} = \rho_{i,j}(H_i) \subset H_j$ for $i \leq j$, then they define $H_{i,k,j} = H_{i,k} \cap (\rho_{k,j})^{-1}(H_{i-1,j}) \subset H_k$ for $i \leq k \leq j$, and finally they define the BD groups as

$$BD_{i,j} = \frac{H_{i,i,j}}{H_{i,i,j-1}} = \frac{H_{i,i+1,j}}{H_{i,i+1,j-1}} = \cdots = \frac{H_{i,j-2,j}}{H_{i,j-2,j-1}} = \frac{H_{i,j-1,j}}{H_{i-1,j-1}} \tag{2}$$

According to [2], a non trivial $BD_{i,j}$ group is meant to show that there is a topological feature (a homology class) that is <u>b</u>orn in step i and <u>d</u>ies in step j, but it does not give a formal proof for this affirmation. All these groups can be computed by a module of Kenzo program, using the spectral sequences defined in [3].

In order to connect the $BD_{i,j}$ groups with the theory developed in [5], which always talks in terms of intervals $I = [i, j)$, we proposed in [1] the alternate notation $BD_{I,k} = \frac{H_{i,k,j}}{H_{i,k,j-1}}$ and we proved that the structure of $BD_{I,k}$ is the same for every $k \in I$. After that, we introduced the V groups defined in [5] as $V_{I,k} = V^+_{I,k}/V^-_{I,k}$, where $V^+_{I,k} = \operatorname{Im} \rho_{i,k} \cap \ker \rho_{k,j}$ and $V^-_{I,k} = (\operatorname{Im} \rho_{i,k} \cap \ker \rho_{k,j-1}) + (\operatorname{Im} \rho_{i-1,k} \cap \ker \rho_{k,j})$. To provide intuition to the spaces $V_{I,k}$, notice that when working with field coefficients, it is proved in [4] that the dimension of $V_{I,k}$ shows how many homology classes are born in step i and die at step j.

In [1], we also introduced a new definition of BD groups for infinite intervals $I = [i, \infty)$, given by $BD_I = \frac{H_{i,m}}{H_{i-1,m}}$, and we proved that this formula is equivalent to $BD_{[i,m+1),k} = \frac{H_{i,k,m+1}}{H_{i,k,m}}$ if we include a last term $H_{m+1} = 0$ and a null homomorphism $\rho_{m,m+1}$ in the equation 1.

Recall that BD groups were defined in [2] for persistent homology with integer coefficients, while V groups were defined in [5] for field coefficients. In [1], we provided a proof that, when working with field coefficients, $BD_{I,k}$ and $V_{I,k}$ are isomorphic. We also proved that, independently on the choice of coefficients, $V^+_{I,k} \subset H_{i,k,j}$ and $V^-_{I,k} \subset H_{i,k,j-1}$. Finally, a proof or a counterexample for the isomorphism between $BD_{I,k}$ and $V_{I,k}$ when working with integer coefficients was left as future work.

After that, we wanted to prove some stability results for BD and V groups, so we needed to extend their definitions for a more general framework. Observe that Equation 1 can be generalized by having a homology group H_i for each $i \in \mathbb{R}$ and linear maps $\rho_{i,j} : H_i \to H_j$ for $i \leq j$, satisfying that $\rho_{i,j} = \rho_{k,j} \circ \rho_{i,k}$ when $i \leq k \leq j$. In this general framework, the author of [5] gave an extended definition for V groups. The extended definition for BD groups was stated by us in [1]. In both cases, we proved in [1] that these last definitions are indeed a good generalization to those used in the more simple framework with only m steps. After that, we proved that, in this general framework, $BD_{I,k}$ and $V_{I,k}$ are also isomorphic when working with field coefficients. We left as future work the proof in case of working with integer coefficients.

Finally, inspired in the theory developed in [6], we gave first steps to stability considering the 1_ε functor, which induces an ε perturbation into persistent homology. We proved how this perturbation affects our BD definition and V groups: by transforming $BD_{(i,j+\varepsilon),k+\varepsilon}$ into $BD_{(i,j),t}$ and $V_{(i,j+\varepsilon),k+\varepsilon}$ into $V_{(i,j),t}$. We do not explore more stability results, but we consider this a good starting point to completely prove that persistent homology is also stable when working with integer coefficients.

In summary, although persistent homology is more difficult to study and apply when using integer coefficients, we show that it is possible to make some connections with the more developed theory for field coefficients, and we think that it is possible to go beyond.

References

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